WEST BENGAL STATE UNIVERSITY

B.Sc. Honours Part-III Examination, 2020

MATHEMATICS

PAPER-MTMA-V

Time Allotted: 2 Hours Full Marks: 50

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

GROUP-A

Marks-20

Answer Question No. 1 and any *one* from the rest

1. Answer any ***three*** questions from the following: $3\times3 = 9$

(a) The sets $C,\ D\subset\mathbb{R}$ are such that C is compact and D is closed, verify which one of $C\cap D$ and $C\cup D$ is a compact set.

(b) Examine if the function $f:[0,1]\to\mathbb{R}$ defined by $f(x)=\sin\frac{\pi}{2x},\ x\neq 0$, $f(0)=0$, is of bounded variation.

(c) Show that the sequence $\{f_n\}$, defined by $f_n(x)=1-x^n,\ 0\leq x\leq 1,\ n\in\mathbb{N}$ is pointwise convergent but not uniformly convergent.

(d) Verify whether the function $f:[0,1]\to\mathbb{R}$ defined by

$$f(x)=x\ ,\ x \text{ is rational}$$
$$=1\ ,\ x \text{ is irrational}$$

is Riemann-integrable.

(e) Find the radius of convergence of the power series $\sum_{n=1}^{\infty} n(-2)^n(x-3)^n$.

(f) Test the convergence of the improper integral $\int_1^{\infty}\frac{dx}{x^p}$, $p>0$ for different values of p.

(g) State the relation between beta function and gamma function. Using it show that $\Gamma(\frac{1}{2})=\sqrt{\pi}$.

(h) Using beta and gamma functions evaluate $\int_0^{\pi/2}\sin^8 x\cos^{12} x\, dx$.

(i) Change the integral $\iint_R (x^2+y^2)^{3/2}\,dx\,dy$ into polar coordinates, where R is the region in the upper half of the coordinate plane bounded by the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$, $a<b$.

2. (a) Prove that every compact subset of R is closed. Is the converse true? Support your answer. 4

(b) Let K be a compact subset of $\mathbb{R}$ and E be an infinite subset of K. Show that E has a limit point in K. 3

(c) Let $f: K \to \mathbb{R}$ be a continuous function on a compact subset K of $\mathbb{R}$. Show that f is uniformly continuous. 4

3. (a) Show that the sequence of functions $\{f_n\}$ is uniformly convergent on $[0,\infty)$, where for all $n\in\mathbb{N}$, 3

$$f_n(x)=xe^{-nx},\ x\geq 0$$

(b) Let $f_n:[a,b]\to\mathbb{R}$ be Riemann integrable for all $n\in\mathbb{N}$ and let the sequence of functions $\{f_n\}$ be uniformly convergent to a function f on $[a,b]$. Show that f is Riemann integrable on $[a,b]$ and 4

$$\int_a^b f(x)\,dx=\lim_{n\to\infty}\int_a^b f_n(x)\,dx.$$

(c) Show that the sequence of functions $\{f_n\}$ converges uniformly on $\mathbb{R}$, where for all $n\in\mathbb{N}$, 2+2

$$f_n(x)=\frac{\sin nx}{\sqrt{n}},\ x\in\mathbb{R}$$

Also show that at $x=0$, $\lim_{n\to\infty} f_n'(x)\neq f'(x)$.

4. (a) If the series $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent on $[a,b]$ and g is a bounded function on $[a,b]$, then prove that the series $\sum_{n=1}^{\infty} g(x)f_n(x)$ is uniformly convergent on $[a,b]$. 3

(b) With proper justification, show that 4

$$\lim_{x\to 0}\sum_{k=2}^{\infty}\frac{\cos kx}{k(k+1)}=\frac{1}{2}$$

(c) A series $\sum_{n=1}^{\infty} f_n(x)$ of differentiable functions on $[0,1]$ is such that for all $n\in\mathbb{N}$ and $x\in[0,1]$, 4

$$\sum_{k=1}^{n} f_k(x)=\frac{\log(1+n^4x^2)}{2n^2}$$

Show that $\frac{d}{dx}\sum_{n=1}^{\infty} f_n(x)=\sum_{n=1}^{\infty} f_n'(x)$ for all $x\in[0,1]$.

5. (a) Use the result that $\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n$ for $|x|<1$ to show that 3+1

$$\log(1+x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} \quad \text{for } |x|\le 1$$

Hence deduce that $\log 2=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$

(b) Show that the integral $\int_0^2 \frac{\sin x}{x^p}\,dx$ is convergent if and only if $p<2$. 4

(c) Use differentiation under integral sign to prove that 3

$$\int_0^{\pi/2}\frac{\log(1+\cos\alpha\cos x)}{\cos x}\,dx=\frac{1}{2}\left(\frac{\pi^2}{4}-\alpha^2\right)$$

6. (a) Let $f:[a,b]\to\mathbb{R}$ be a bounded monotone function. Show that f is Riemann integrable on $[a,b]$. 4

(b) Let $f, g:[a,b]\to\mathbb{R}$ be bounded functions such that $f=g$ except at a finite number of points in $[a,b]$. If f is Riemann integrable over $[a,b]$, then show that g is Riemann integrable over $[a,b]$ and $\int_a^b f(x)\,dx=\int_a^b g(x)\,dx$. 4

(c) Justify with an example that, a bounded function on a closed interval which is not Riemann integrable may have a primitive. 3

7. (a) Show that $\frac{1}{2}<\int_0^1\frac{1}{\sqrt{4-x^2+x^3}}\,dx<\frac{\pi}{6}$. 4

(b) For $0<a<b<\infty$, prove that $\left|\int_a^b\frac{\sin x}{x}\,dx\right|\le\frac{2}{a}$. 3

(c) Evaluate $\underline{\int_0^1} f\,dx$ and $\overline{\int_0^1} f\,dx$ and hence examine the integrability of f on [0, 1] where 4

$$f(x)=\begin{cases} x+x^3 & ; \quad x\in\mathbb{Q}\cap[0,1] \\ x^2+x^3 & ; \quad x\in[0,1]-\mathbb{Q}\end{cases}$$

8. (a) If $f:[a,b]\to\mathbb{R}$ is a bounded Riemann integrable function on $[a,b]$ and 4

$$F(x)=\int_a^x f(t)\,dt \quad \text{for all } x\in[a,b],$$

then show that F is of bounded variation on $[a,b]$.

(b) Let $f,\, g:[0,1]\to \mathbb{R}$ be defined by $f(x)=x^2$ and $g(x)=\begin{cases} x^2\sin\frac{1}{x} &, \ 0<x\le 1 \\ 0 &, \ x=0 \end{cases}$ 4

Show that the plane curve $\gamma(x)=(f(x),\, g(x))$ is rectifiable on [0, 1].

(c) Let $a>0$ be a constant. Find the length of one arch of the cycloid $x=a(\theta-\sin\theta),\quad y=a(1-\cos\theta)$. 3

9. (a) With proper justification, apply Mean Value Theorem for functions of two variables to the function $f(x,\, y)=\sin x\cos y$ to show that there is a $\theta\in(0,1)$ for which $\frac{3}{4}=\frac{\pi}{3}\cos\frac{\pi\theta}{3}\cos\frac{\pi\theta}{6}-\frac{\pi}{6}\sin\frac{\pi\theta}{3}\sin\frac{\pi\theta}{6}$. 4

(b) Obtain Taylor's expansion of the function $\log\left(\frac{x+y}{2}\right),\ x>0,\ y>0$ about the point (1, 1) up to the terms of degree three. 3

(c) Use Lagrange's method of undetermined multiplier to find the stationary value of $u=a^3x^2+b^3y^2+c^3z^2$ where $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$. 4

10.(a) Find the Fourier series of the function 4+2

$$f(x)=x-\pi\ ,\ -\pi\le x<0$$
$$=\pi-x\ ,\ \ 0\le x\le\pi$$

in the interval $[-\pi,\, \pi]$. Find the sum of the series at $x=0$ and hence deduce the identity $\frac{\pi^2}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\cdots$.

(b) Evaluate the integral $\iint_R \frac{2x^2+y^2}{xy}\,dx\,dy$, where R is the region in the positive quadrant of the *xy*-plane bounded by circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$ and the parabolas $y^2=4cx$ and $y^2=4dx,\ 0<a<b,\ 0<c<d$ by taking the substitution $x^2+y^2=u,\ \frac{y^2}{4x}=v$. 5

GROUP-B

(Marks-15)

Answer any *one* question from the following 15×1 = 15

11.(a) Define (i) open ball (ii) open set in a metric space. Prove that if $B(a_1,\, r_1)$ and $B(a_2,\, r_2)$ are two open balls then for any $x\in B(a_1,\, r_1)\cap B(a_2,\, r_2)$ there exists $r>0$ such that $B(x,\, r)\subset B(a_1,\, r_1)\cap B(a_2,\, r_2)$. 1+1+3

(b) Let (X, d) be a metric space and $d^*: X \times X \to R$ be defined by 4+1

$$d^*(x, y) = \frac{d(x, y)}{1+d(x, y)}, \quad x, y \in X$$

Prove that (X, d^*) is a metric space. Examine if the metric d^* is bounded.

(c) Let (X, d) be a metric space. Prove that a set $V \subset X$ is open if and only if its complement $X - V$ is closed. 5

12.(a) Let (X, d) be a metric space and $A, B \subset X$. Prove that $(A \cup B)' = A' \cup B'$ where A' denotes the derived set of A. Does the result hold if $\cup$ is replaced by $\cap$? Support your answer. 3+1

(b) Define a Cauchy sequence in a metric space. Let (X, d) denote a discrete metric space, where $d: X \times X \longrightarrow \mathbb{R}$ defined by $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$, $x, y \in X$. Show that in a discrete metric space a sequence is a Cauchy sequence if and only if it is eventually constant. 1+4

(c) When is a metric space called complete? Show that the discrete metric space is a complete metric space. 1+2

(d) Define completion of a metric space. What is the completion of the metric space consisting of the rational numbers? 2+1

GROUP-C

(Marks-15)

Answer any *one* question from the following 15×1 = 15

13.(a) If $f(z) = u(x, y) + iv(x, y)$ where $u(x, y)$ and $v(x, y)$ are both real-valued functions, be defined on a region G except possibly at $z_0 = x_0 + iy_0$ then prove that $\lim\limits_{z \to z_0} f(z) = a + ib$ if and only if $\lim\limits_{\substack{x \to x_0 \\ y \to y_0}} u(x, y) = a$ and $\lim\limits_{\substack{x \to x_0 \\ y \to y_0}} v(x, y) = b$. 4

(b) If f be an analytic function defined on $E \subseteq \mathbb{C}$ such that $|f|$ is constant on E then prove that f is a constant function on E. 3

(c) Let $f: \mathbb{C} \longrightarrow \mathbb{C}$ be defined by 5

$$f(x+iy) = \begin{cases} \dfrac{x^2 y^5 (x+iy)}{x^4 + y^{10}}, & x+iy \neq 0 \\ 0, & x+iy = 0 \end{cases}$$

Prove that f satisfies the CR equations at the origin but f is not differentiable at the origin.

(d) If $f: D \longrightarrow \mathbb{C}$ be continuous at $z_0 \in D$ and $\{z_n\}$ be a sequence in D converging to z_0 then prove that $\{f(z_n)\}$ converges to $f(z_0)$. 3

14.(a) Let $f: G \longrightarrow \mathbb{C}$ where $f(x+iy) = u(x, y) + iv(x, y)$ be defined on a region G and f be differentiable at $z_0 = x_0 + iy_0$. Prove that the functions $u(x, y)$ and $v(x, y)$ are differentiable at (x_0, y_0) and the following equations are satisfied there at 5

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

(b) Prove that the function $f(z) = \bar{z}$ on $\mathbb{C}$ is not differentiable at any point in $\mathbb{C}$. 3

(c) Show that $u(x, y) = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic and find the analytic function f of which $u(x, y)$ is the real part. 4

(d) Show that the function $f(x+iy) = x^2 + iy^2$ is nowhere analytic. 3

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